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## A Determination of the Earth's Gravity Field in Spheroidal Coordinates

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A DETERMINATION OF THE EARTH'S GRAVITY FIELD  
IN SPHEROIDAL COORDINATES

by

M. Spencer Hamilton, Jr.

A thesis submitted in partial fulfillment  
of the requirements for the degree

of

MASTER OF SCIENCE

in

Mathematics

Approved:

UTAH STATE UNIVERSITY  
Logan, Utah

1961

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## INTRODUCTION

The earth's gravity field  $\vec{G}^*$  at a point P in the region surrounding the earth's surface is defined as the force acting on a unit mass concentrated at P. This is a force resulting from two components: (1)  $\vec{G}_1$  due to the gravitational attraction of the earth's mass, and (2)  $\vec{G}_2$  due to the earth's rotation.

As a result of Newton's law of gravitation,  $\vec{G}_1$  can be written in integral form as follows:

$$\vec{G}_1 = k \iiint_V \frac{\vec{r} dm}{r^3} \quad (1)$$

where  $\vec{r} = \vec{PQ}$ ,  $r = |\vec{r}|$ , Q is a point which ranges over the region V bounded by the earth's surface, k is the gravitational constant,  $dm = \rho dv$ , and  $\rho$  is the density at Q.

The density function,  $\rho$ , is not known, and so equation (1) is not suitable for the determination of  $\vec{G}_1$ . However, it can be shown that the earth's gravity field is conservative, and therefore, there exists a scalar function U such that  $\vec{G}$  is determined by the gradient of U; that is

$$\vec{G} = -\nabla U \quad (2)$$

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\*Vectors will be indicated throughout by placing an arrow above the letter.

The function  $U$  is called the total potential and can be written as the sum of two functions,  $U_1$  and  $U_2$ , where  $U_1$  is the potential corresponding to  $\vec{G}_1$ , and  $U_2$  is that associated with  $\vec{G}_2$ .

An important property of the function  $U_1$  is that it satisfies Laplace's differential equation, that is

$$\nabla^2 U_1 = 0 \quad (3)$$

Various mathematical expressions have been determined which represent the gravity field of the earth with varying degrees of accuracy. The main problem in obtaining these results is that of solving Laplace's equation with appropriate boundary conditions to determine  $U_1$ . The boundary conditions involve the shape of the surface of the earth. By assuming the earth's surface to be ellipsoidal and using spherical coordinates, a reasonably accurate solution in infinite series form can be found. Based on the first few terms of this series and on certain accepted measurements related to the size and shape of the earth's surface, the International Gravity Formula is obtained:

$$|\vec{G}| = g = g_0(1 + 0.0052884 \sin^2 \theta - 0.0000059 \sin^2 2\theta) \quad (4)$$

where  $\theta$  is the geographical latitude and  $g_0$  represents the measured acceleration of gravity at the equator.

## STATEMENT OF PROBLEM AND PROCEDURE

The surface of the earth will be assumed to be an oblate spheroid (ellipse revolved about its minor axis), and, therefore, it is natural to search for a solution in terms of oblate spheroidal coordinates. Laplace's differential equation will be expressed in spheroidal coordinates and a solution for  $U_1$  will be found by the method of separation of variables, and  $U_1$  will be required to have continuous second order derivatives and vanish at infinity. The total potential,  $U = U_1 + U_2$ , expressed in spheroidal coordinates, will then be determined such that it is constant on the earth's surface. This will result in a solution in closed form. Then  $\nabla U$  will yield an expression for  $\vec{G}$  in non-infinite series form.

As a check,  $|\vec{G}|$  as found above, will be compared with that given by the International Gravity Formula.

## SOLUTION OF PROBLEM

### Transformation From Rectangular to Spheroidal Coordinates

Let a point, P, have rectangular coordinates  $(x, y, z)$ , and let  $r_1$  and  $r_2$  be the distances from  $(c, 0, 0)$  to  $(x, 0, z)$  and from  $(-c, 0, 0)$  to  $(x, 0, z)$  respectively, where  $c > 0$ . Define  $u$  and  $v$  as follows:

$$u = \frac{r_1 + r_2}{2c}, \quad u \geq 1 \quad (5)$$

$$v = \frac{r_1 - r_2}{2c}, \quad -1 \leq v \leq 1$$

The curves,  $u = \text{constant}$ ,  $y = 0$ , represent a family of confocal ellipses in the  $xz$ -plane, and the curves,  $v = \text{constant}$ ,  $y = 0$ , a family of confocal hyperbolas in the  $xz$ -plane. Now let the above system of confocal ellipses and hyperbolas in the  $xz$ -plane be rotated about the  $z$  axis, and let  $\phi$  be the angle of rotation about this axis measured from the positive  $X$ -axis. The coordinates,  $u, v, \phi$  thus defined are called oblate spheroidal coordinates and form an orthogonal system. The surfaces  $u = \text{constant}$  are oblate spheroids; the surfaces  $v = \text{constant}$  are hyperboloids of one sheet; the surfaces  $\phi = \text{constant}$  are planes through the origin.



It can be shown that the transformation from oblate spheroidal coordinates to rectangular coordinates is given by the following equations:

$$\begin{aligned}x &= cuv \sin \varphi, \\y &= cuv \cos \varphi, \\z &= c \sqrt{(u^2-1)(1-v^2)},\end{aligned}\tag{6}$$

where  $u \geq 1$ ,  $-1 \leq v \leq 1$ ,  $0 \leq \varphi < 2\pi$

Laplace's differential equation in oblate spheroidal coordinates becomes

$$\begin{aligned}\nabla^2 U_1 &= \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u} \left( \frac{h_2 h_3}{h_1} \frac{\partial U_1}{\partial u} \right) + \frac{\partial}{\partial v} \left( \frac{h_1 h_3}{h_2} \frac{\partial U_1}{\partial v} \right) \\&+ \frac{\partial}{\partial \varphi} \left( \frac{h_1 h_2}{h_3} \frac{\partial U_1}{\partial \varphi} \right) = 0\end{aligned}\tag{7}$$

where  $h_1^2 = c^2 \frac{u^2 - v^2}{u^2 - 1}$ ;  $h_2^2 = c^2 \frac{u^2 - v^2}{1 - v^2}$ ,  $h_3 = cuv$ .

Substituting the expressions for  $h_1$ ,  $h_2$ , and  $h_3$  in equation (7), and noting that the potential  $U_1$  is assumed to be independent of  $\varphi$  because of the axial symmetry of the spheroid, the result is

$$\frac{\partial}{\partial u} \left( \sqrt{\frac{u^2-1}{1-v^2}} uv \frac{\partial U_1}{\partial u} \right) + \frac{\partial}{\partial v} \left( uv \sqrt{\frac{1-v^2}{u^2-1}} \frac{\partial U_1}{\partial v} \right) = 0\tag{8}$$

where  $U_1 = U_1(u, v)$ .

For purposes of simplification, it is desirable to make the following transformation of variables:

$$\begin{aligned}u &= \cosh x, \quad x \geq 0 \\v &= \sin y, \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}.\end{aligned}\tag{9}$$



With this change in variables equation (8) becomes

$$\frac{\partial}{\partial x} \left( \cosh x \sin y \frac{\partial U_1}{\partial x} \right) + \frac{\partial}{\partial y} \left( \cosh x \sin y \frac{\partial U_1}{\partial y} \right) = 0 \quad (10)$$

Solutions of equation (10) will be required to have continuous second order derivatives and vanish at infinity.

### Solution of Laplace's Equation in Spheroidal Coordinates

Using the method of separation of variables a solution of equation (10) will be sought. Assume a solution of the form

$$U_1 = F(x) H(y). \quad (11)$$

Substitute this in equation (10), and the result is

$$\frac{1}{F \cosh x} \frac{d}{dx} \left( \cosh x \frac{dF}{dx} \right) = - \frac{1}{H \sin y} \frac{d}{dy} \left( \sin y \frac{dH}{dy} \right) \quad (12)$$

In order for equation (12) to hold, both members must be equal to the same constant. This results in

$$\begin{aligned} \frac{1}{F \cosh x} \frac{d}{dx} \left( \cosh x \frac{dF}{dx} \right) = \\ - \frac{1}{H \sin y} \frac{d}{dy} \left( \sin y \frac{dH}{dy} \right) = C. \end{aligned} \quad (13)$$

One of the equations from the system (13) is

$$\frac{d}{dy} \left( \sin y \frac{dH}{dy} \right) + CH \sin y = 0 \quad (14)$$

This is Legendre's differential equation and it can have a non-zero solution with continuous first order derivatives in the interval  $-1 \leq x \leq 1$  only if\*

$$C = n(n+1) \quad n = 0, 1, 2, 3, \dots \quad (15)$$

Equation (14) can now be written

$$\frac{d}{dy} \left( \sin y \frac{dH}{dy} \right) + n(n+1) H \sin y = 0 \quad (16)$$

Solutions of equation (16) are

$$H(y) = B_1 P_n(\cos y) + B_2 Q_n(\cos y), \quad (17)$$

where  $P_n$  is a Legendre polynomial,  $Q_n$  is a Legendre function of the second kind, and  $B_1$  and  $B_2$  are arbitrary constants. Since Legendre functions of the second kind become infinite for  $x = \pm 1$ , the constant  $B_2$  is taken to be zero.

The other equation from (13) is

$$\frac{d}{dx} \left( \cosh x \frac{dF}{dx} \right) - n(n+1) F \cosh x = 0 \quad (18)$$

This is also Legendre's differential equation and the general solution is

$$F(x) = C_1 P_n(i \sinh x) + C_2 Q_n(i \sinh x). \quad (19)$$

It will appear later that for points outside the earth's surface  $\sinh x > 1$ . Since  $|P_n(i \sinh x)|$  becomes large with large  $x$ , and  $Q_n(i \sinh x)$  approaches zero,  $C_1$  will

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\*Churchill, Ruel V. Fourier Series and Boundary Value Problems. New York: McGraw-Hill Publishing Co., 1941. p. 186.

be set equal to zero. From equation (11), (17), and (19) the desired solution of equation (10) is

$$U_1 = A Q_n(i \sinh x) P_n(\cos y), \quad (20)$$

where  $A$  is an arbitrary constant.

Consider a solution of the following type,

$$U_1 = \sum_{n=0}^{\infty} A_n Q_n(i \sinh x) P_n(\cos y), \quad (21)$$

where the  $A_n$ 's are constants to be determined by the boundary condition.

### Determination of $U_2$

To obtain the potential  $U_2$ , due to the earth's rotation, we note that for any body attached to the earth, the rotation produces an acceleration perpendicular to the axis of rotation. If  $|\vec{S}|$  is the perpendicular distance from the axis of rotation to the body and  $\vec{\omega}$  is the angular rotational velocity of the earth, we write

$$\nabla U_2 = \vec{a} = \omega^2 \vec{S}. \quad (22)$$

Integrating equation (22) gives

$$U_2 = \frac{1}{2} \omega^2 S^2. \quad (23)$$

Using equations (6) and (9) the above expression may be written in spheroidal coordinates,

$$U_2 = \frac{1}{2} \omega^2 c^2 \cosh^2 x \sin^2 y. \quad (24)$$

# Application of Boundary Condition to Determine U

Adding equations (21) and (24) gives an expression for the total potential of the earth,

$$U = \sum_{n=0}^{\infty} A_n Q_n(i \sinh x) P_n(\cos y) + \frac{1}{2} w^2 c^2 \cosh^2 x \sin^2 y. \quad (25)$$

Expression (25) will be made to satisfy the boundary condition that the spheroid representing the surface of the earth is an equipotential surface; that is

$$\lim_{x \rightarrow x_0} U = K, \quad (26)$$

where  $x_0$  is the value of  $x$  on the earth's surface and  $K$  is a constant. Thus we have

$$K = \sum_{n=0}^{\infty} A_n Q_n(i \sinh x_0) P_n(\cos y) + \frac{1}{2} w^2 c^2 \cosh^2 x_0 \sin^2 y. \quad (27)$$

The following properties of Legendre polynomials will be useful in determining the constants  $A_n$ :

$$\begin{aligned} \int_0^{\pi} P_n(\cos y) P_m(\cos y) d(\cos y) &= 0 \\ \text{if } n \neq m, \text{ and} & \\ \int_0^{\pi} [P_n(\cos y)]^2 d(\cos y) &= \frac{2}{2n+1} \quad n=0,1,2,\dots \end{aligned} \quad (28)$$



$$\int_0^\pi P_n(\cos y) d(\cos y) = \begin{cases} 0 & n = 1, 2, 3, \dots \\ 2 & n = 0 \end{cases}$$

$$\int_0^\pi \sin^2 y P_n(\cos y) d(\cos y) = \begin{cases} 0 & n = 1, 3, 4, 5, \dots \\ 4/3 & n = 0 \\ -4/15 & n = 2 \end{cases} \quad (29)$$

Multiplying both sides of equation (27) by  $P_n(\cos y) d(\cos y)$  and integrating, and making use of equation (28) gives

$$K \int_0^\pi P_n(\cos y) d(\cos y) = \frac{2}{2n+1} A_n Q_n(i \sinh x_0) + 1/2 w^2 c^2 \cosh^2 x_0 \int_0^\pi \sin^2 y P_n(\cos y) d(\cos y). \quad (30)$$

Substituting the results from equations (29) in (30) and solving for the constants  $A_n$  yields,

$$A_0 = \frac{K - 1/3 w^2 c^2 \cosh^2 x_0}{Q_0(i \sinh x_0)}$$

$$A_2 = \frac{w^2 c^2 \cosh^2 x_0}{3Q_2(i \sinh x_0)} \quad (31)$$

$$A_n = 0 \quad \text{for } n = \begin{cases} 2m - 1 \\ 2m + 2 \end{cases} \quad m = 1, 2, 3, \dots$$

Substitution of the values for  $A_n$  given in equation (31) into equation (25) gives

$$U = \frac{K - 1/3 w^2 c^2 \cosh^2 x_0}{Q_0(i \sinh x_0)} Q_0(i \sinh x) P_0(\cos y)$$



$$\begin{aligned}
& + \frac{w^2 c^2 \cosh^2 x_0}{3 Q_2(i \sinh x_0)} Q_2(i \sinh x) P_2(\cos y) \\
& + 1/2 w^2 c^2 \cosh^2 x \sin^2 y.
\end{aligned} \tag{32}$$

Expressions for  $Q_0$ ,  $Q_2$ ,  $P_0$ ,  $P_2$  in terms of the original variables  $u$  and  $v$  are as follows:

$$P_0(\cos y) = 1 \tag{33}$$

$$P_2(\cos y) = 1/2 (2 - 3 \sin^2 y) = 1/2 (2 - 3v^2) \tag{34}$$

$$\begin{aligned}
Q_0(i \sinh x) &= Q_0(i \sqrt{u^2 - 1}) = \frac{1}{i \sqrt{u^2 - 1}} \\
&- \frac{1}{3i(\sqrt{u^2 - 1})^3} + \frac{1}{5i(\sqrt{u^2 - 1})^5} - \dots
\end{aligned} \tag{35}$$

$$\begin{aligned}
Q_2(i \sinh x) &= Q_2(i \sqrt{u^2 - 1}) = (1 - 3/2 u^2) Q_0(i \sqrt{u^2 - 1}) \\
&- 3/2 i \sqrt{u^2 - 1}.
\end{aligned} \tag{36}$$

Using equations (33), (34), and (36) in (32) the variables  $u$  and  $v$  are reintroduced and the result is

$$\begin{aligned}
U(u, v) &= \frac{K - 1/3 w^2 c^2 u_0^2}{Q_0(i \sqrt{u_0^2 - 1})} Q_0(i \sqrt{u^2 - 1}) \\
&+ \frac{w^2 c^2 u_0^2 (2 - 3v^2) \left[ (1 - 3/2 u^2) Q_0(i \sqrt{u^2 - 1}) - 3/2 i \sqrt{u^2 - 1} \right]}{6 \left[ (1 - 3/2 u_0^2) Q_0(i \sqrt{u_0^2 - 1}) - 3/2 i \sqrt{u_0^2 - 1} \right]} \\
&+ 1/2 w^2 c^2 u^2 v^2.
\end{aligned} \tag{37}$$

The vector field  $\vec{G}$  can now be found by taking the gradient of  $U(u, v)$ .

This will be accomplished by using the following formula which gives the gradient in spheroidal coordinates:

$$\nabla U(u, v) = \frac{1}{c} \sqrt{\frac{u^2-1}{u^2-v^2}} \frac{\partial U}{\partial u} \vec{i}_u + \frac{1}{c} \sqrt{\frac{1-v^2}{u^2-v^2}} \frac{\partial U}{\partial v} \vec{i}_v, \quad (38)$$

where  $\vec{i}_u$  and  $\vec{i}_v$  are unit vectors at the point P normal to the surfaces  $u = \text{constant}$  and  $v = \text{constant}$ , respectively, which pass through the point P. Therefore, the gradient of  $U(u, v)$  is

$$\begin{aligned} \vec{G} = & \left\{ \frac{K - 1/3 w^2 c^2 u_0^2}{iu \sqrt{u^2-1} Q_0(i \sqrt{u_0^2-1})} \right. \\ & - \frac{w^2 c^2 u_0^2 \left[ 3u Q_0(i \sqrt{u^2-1}) + \frac{1-3u^2}{iu \sqrt{u^2-1}} \right]}{6 \left[ (3/2 u_0^2-1) Q_0(i \sqrt{u_0^2-1}) + 3/2 i \sqrt{u_0^2-1} \right]} (2-3v^2) \\ & \left. - w^2 c^2 uv^2 \right\} \frac{1}{c} \sqrt{\frac{u^2-1}{u^2-v^2}} \vec{i}_u \\ & + \left\{ \frac{w^2 c^2 u_0^2 v \left[ (3/2 u^2-1) Q_0(i \sqrt{u^2-1}) + 3/2 i \sqrt{u^2-1} \right]}{(3/2 u_0^2-1) Q_0(i \sqrt{u_0^2-1}) + 3/2 i \sqrt{u_0^2-1}} \right. \\ & \left. - w^2 c^2 u^2 v \right\} \frac{1}{c} \sqrt{\frac{1-v^2}{u^2-v^2}} \vec{i}_v, \quad (39) \end{aligned}$$

since it follows from equation (35) that,

$$\frac{\partial}{\partial u} \left[ Q_0(i \sqrt{u^2-1}) \right] = - \frac{1}{iu \sqrt{u^2-1}}. \quad (40)$$

In order to determine the constant K, let  $u = u_0$  and  $v = 1$ . These are the coordinates of a point on the

earth's equator. At the equator  $|\vec{G}| = |\vec{G}_0| = g_0$ .

The magnitude of equation (39) then becomes

$$g_0 = 1/c \left\{ \frac{K - 1/3 w^2 c^2 u_0^2}{i u_0 \sqrt{u_0^2 - 1} Q_0(i \sqrt{u_0^2 - 1})} + \frac{w^2 c^2 u_0^2 \left[ 3 u_0 Q_0(i \sqrt{u_0^2 - 1}) + \frac{1 - 3 u_0^2}{i u_0 \sqrt{u_0^2 - 1}} \right]}{6 \left[ (3/2 u_0^2 - 1) Q_0(i \sqrt{u_0^2 - 1}) + i 3/2 \sqrt{u_0^2 - 1} \right]} - w^2 c^2 u_0^2 \right\}. \quad (41)$$

Solution of equation (41) for K gives,

$$K = \left\{ g_0 c - \frac{w^2 c^2 u_0^2 \left[ 3 u_0 Q_0(i \sqrt{u_0^2 - 1}) + \frac{1 - 3 u_0^2}{i u_0 \sqrt{u_0^2 - 1}} \right]}{6 \left[ (3/2 u_0^2 - 1) Q_0(i \sqrt{u_0^2 - 1}) + 3/2 i \sqrt{u_0^2 - 1} \right]} + w^2 c^2 u_0^2 \right\} Q_0(i \sqrt{u_0^2 - 1}) i u_0 \sqrt{u_0^2 - 1} + 1/3 w^2 c^2 u_0^2. \quad (42)$$

Upon substitution of equation (42) into equation (39) the result is

$$\vec{G} = \left\{ \frac{g_0 - \frac{w^2 c u_0 \left[ 3 u_0^2 Q_0(i \sqrt{u_0^2 - 1}) + \frac{1 - 3 u_0^2}{i \sqrt{u_0^2 - 1}} \right]}{6 \left[ (3/2 u_0^2 - 1) Q_0(i \sqrt{u_0^2 - 1}) + 3/2 i \sqrt{u_0^2 - 1} \right]}}{u \sqrt{u^2 - 1}} - \frac{w^2 c u_0}{u \sqrt{u^2 - 1}} \right\} u_0 \sqrt{u_0^2 - 1}$$

$$\begin{aligned}
& - \left[ \frac{w^2 c u_0^2 \left( 3u Q_0(i\sqrt{u^2-1}) + \frac{1-3u^2}{iu\sqrt{u^2-1}} \right)}{6 \left( (3/2 u_0^2-1) Q_0(i\sqrt{u_0^2-1}) + 3/2 i\sqrt{u_0^2-1} \right)} \right]^{(2-3v^2)} \\
& - w^2 c u v^2 \left\{ \sqrt{\frac{u^2-1}{u^2-v^2}} \vec{i}_u \right. \\
& + \left. \frac{w^2 c u_0^2 v \left[ (3/2 u^2-1) Q_0(i\sqrt{u^2-1}) + 3/2 i\sqrt{u^2-1} \right]}{(3/2 u_0^2-1) Q_0(i\sqrt{u_0^2-1}) + 3/2 i\sqrt{u_0^2-1}} \right. \\
& \left. - w^2 c u^2 v \left\{ \sqrt{\frac{1-v^2}{u^2-v^2}} \vec{i}_v \right. \right. \quad (43)
\end{aligned}$$

This is an expression for the earth's gravity field at any point outside the earth's surface, where the point is assumed to be rotating with the earth.

#### Gravity Field on the Earth's Surface

An expression for the gravity field on the earth's surface can be obtained by setting  $u = u_0$  in equation (43). The result after some simplification is

$$\begin{aligned}
\vec{G} = & \sqrt{\frac{u_0^2-1}{u_0^2-v^2}} \left\{ g_0 - (1-v^2) w^2 c u_0 \right. \\
& \left. \cdot \left[ \frac{3u_0^2 Q_0(i\sqrt{u_0^2-1}) + 1-3u_0^2/\sqrt{u_0^2-1}}{2 \left[ (3/2 u_0^2-1) Q_0(i\sqrt{u_0^2-1}) + 3/2 (i\sqrt{u_0^2-1}) \right]} - 1 \right] \right\} \vec{i}_u \quad (44)
\end{aligned}$$



The constants  $u_0$ ,  $c$ , and  $w$  will be assigned values as follows:\*

$$u_0 = 12.1963282$$

$$c = 522.97608714 \times 10^5 \text{ cm.}$$

$$w = 7.292115147 \times 10^{-5} \text{ sec}^{-1}.$$

Using these values we have

$$w^2 c u_0 \left[ \frac{3u_0^2 Q_0(i\sqrt{u_0^2-1}) + \frac{1-3u_0^2}{i\sqrt{u_0^2-1}}}{2 \left( (3/2u_0^2-1)Q_0(i\sqrt{u_0^2-1}) + 3/2 i\sqrt{u_0^2-1} \right)} - 1 \right] = -8.49442.$$

Upon substitution of this value in equation (44), the result is

$$G = \sqrt{\frac{u_0^2-1}{u_0^2-v^2}} \left[ g_0 + 8.49442(1-v^2) \right] \vec{i}_u, \quad (45)$$

where  $g_0 = 978.049 \text{ cm. sec}^{-2}$ .

This is a mathematical expression for the earth's gravity field at any point on the earth's surface. The magnitude of equation (45) which gives the magnitude of the

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\*Landsberg, H. E., Ed. Advances in Geophysics. New York: Academic Press, 1958. p. 103.



acceleration of gravity on the earth's surface is

$$|\vec{G}| = g = \sqrt{\frac{u_o^2 - 1}{u_o^2 - v^2}} \left[ g_o + 8.49442 (1 - v^2) \right] . \quad (46)$$

## COMPARISON OF SOLUTIONS

In order to compare the expressions for the magnitude of the earth's gravity field as obtained in this paper with that as given by the International Gravity Formula, a transformation will be made in equation (46) giving the variable  $v$  in terms of geocentric latitude. Then the geographic latitude in the International Gravity Formula will be transformed to terms of geocentric latitude. Finally, the results will be compared.

It can be shown from equations (6) and

$$R \cos \Theta' = cuv$$

where  $R$  is the distance from the center of the earth to a point on the earth's surface whose geocentric latitude is  $\Theta'$ , that

$$v^2 = \frac{(u_o^2 - 1) \cos^2 \Theta'}{u_o^2 - \cos^2 \Theta'} \quad (47)$$

Substituting equation (47) in equation (46) gives

$$g = \left[ \frac{1 + \frac{\sin^2 \Theta'}{u_o^2 - 1}}{1 + \frac{(2u_o^2 - 1) \sin^2 \Theta'}{(u_o^2 - 1)^2}} \right]^{1/2} \left[ g_o + \right.$$

$$\left. 8.49442 \frac{u_o^2 \sin^2 \theta'}{u_o^2 - 1 + \sin^2 \theta'} \right] . \quad (48)$$

Upon expanding equation (48) the result is

$$\begin{aligned} g = g_o & \left\{ 1 + \left[ \frac{g_o + 17.17684 u_o^2}{2g_o(u_o^2 - 1)} - \frac{2u_o^2 - 1}{2(u_o^2 - 1)^2} \right] \sin^2 \theta' \right. \\ & + \left[ - \frac{8.49442 u_o^2}{2g_o(u_o^2 - 1)^2} - \frac{8.49442 u_o^2 (2u_o^2 - 1)}{2g_o(u_o^2 - 1)^3} \right. \\ & \left. \left. + \frac{3(2u_o^2 - 1)^2}{8(u_o^2 - 1)^4} - \frac{2u_o^2 - 1}{4(u_o^2 - 1)^3} - \frac{1}{8(u_o^2 - 1)^2} \right] \right. \\ & \left. \cdot \sin^4 \theta' + \dots \right\} . \quad (49) \end{aligned}$$

After evaluating the coefficients, equation (49) becomes

$$\begin{aligned} g = g_o & \left[ 1 + 0.0053369 \sin^2 \theta' \right. \\ & \left. - 0.0000485 \sin^4 \theta' + \dots \right] . \quad (50) \end{aligned}$$

Expressing the International Gravity Formula in terms of geocentric latitude\* the result is

$$\begin{aligned} g = g_o & \left[ 1 + 0.0053357 \sin^2 \theta' \right. \\ & \left. - 0.0000415 \sin^4 \theta' + \dots \right] . \quad (51) \end{aligned}$$

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\*Heiskanen, W. A., and F. A. Vening Meinesz. The Earth and its Gravity Field. New York: McGraw-Hill Publishing Co., 1958. p. 49.

# SUMMARY

The expression obtained in this paper for the earth's gravity field at any point assumed to be rotating with the earth is given by equation (43) and with the constants evaluated becomes

$$\begin{aligned} \vec{G} = & \left[ 148.250 \left( \frac{g_0 + 5.09261}{u \sqrt{u^2 - 1}} \right) \right. \\ & - 93407.3(2-3v^2) \left( i3uQ_0(i\sqrt{u^2-1}) + \frac{1-3u^2}{u\sqrt{u^2-1}} \right) \\ & \left. - 0.278092uv^2 \right] \sqrt{\frac{u^2-1}{u^2-v^2}} \vec{i}_u \\ & + \left[ 93407.3 \left( (3/2 u^2-1) iQ_0(i\sqrt{u^2-1}) - 3/2 \sqrt{u^2-1} \right) \right. \\ & \left. - 0.278092u^2v \right] \sqrt{\frac{1-v^2}{u^2-v^2}} \vec{i}_v, \end{aligned} \quad (52)$$

where  $u$  and  $v$  are spheroidal coordinates of the point at which  $\vec{G}$  is to be evaluated and

$$g_0 = 978.049 \text{ cm. sec}^{-2}.$$

$Q_0$  is a Legendre function of the second kind of order zero.

For a point on the earth's surface, the expression for the gravity field as given in equation (45) is

$$\vec{G} = \sqrt{\frac{u_0^2-1}{u_0^2-v^2}} \left[ g_0 + 8.49442 (1-v^2) \right] \vec{i}_u. \quad (53)$$



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